

Recap: Some properties of multivariate normal.

$$\underline{y} = \underline{x}\underline{\beta} + \underline{e}, \quad \underline{e} \sim N(\underline{0}, \sigma^2 \underline{V})$$

$\underline{V} = \underline{I}$ when $\underline{V} \neq \underline{I}$ we have seen that

$$\underline{V}^{-1/2} \underline{y} = \underline{V}^{-1/2} \underline{x} \underline{\beta} + \underline{V}^{-1/2} \underline{e}, \quad \underline{V}^{-1/2} \underline{e} \sim N(\underline{0}, \sigma^2 \underline{I})$$

$\underline{x} \sim N(\underline{\mu}, \underline{V})$, $\underline{y} = \underline{a} + \underline{B} \underline{x}$, \underline{a} is $q \times 1$ vector

and \underline{B} is $q \times p$ then, $\underline{y} \sim N_q(\underline{a} + \underline{B} \underline{\mu}, \underline{B} \underline{V} \underline{B}^T)$.

Result: $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$, $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$

$$\underline{V} = \begin{bmatrix} \underline{V}_{11} & \underline{V}_{12} \\ \underline{V}_{21} & \underline{V}_{22} \end{bmatrix}$$

$p_1 \times p_1$ $p_1 \times p_2$
 $p_2 \times p_1$ $p_2 \times p_2$

$$\underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix}$$

$p_1 \times 1$
 $p_2 \times 1$

$$\underline{x}_2 | \underline{x}_1 \sim N_{p_2}(\underline{\mu}_2 + \underline{V}_{21} \underline{V}_{11}^{-1} (\underline{x}_1 - \underline{\mu}_1), \underline{V}_{22} - \underline{V}_{21} \underline{V}_{11}^{-1} \underline{V}_{12})$$

Result: $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$ of the above form

$$\underline{x}_1 \sim N_{p_1}(\underline{\mu}_1, \underline{V}_{11}), \quad \underline{x}_2 \sim N_{p_2}(\underline{\mu}_2, \underline{V}_{22})$$

*

Result: $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$, $\underline{y}_1 = \underline{a}_1 + \underline{B}_1 \underline{x}$ and

$\underline{y}_2 = \underline{a}_2 + \underline{B}_2 \underline{x}$ then \underline{y}_1 and \underline{y}_2 are independent

iff $\underline{B}_1 \underline{V} \underline{B}_2^T = \underline{0}$

$$\textcircled{6} \quad \underline{y} = \hat{\underline{y}} + \hat{\underline{e}} = \underline{P}\underline{y} + (\underline{I} - \underline{P})\underline{y}$$

claim: $\hat{\underline{y}}$ and $\hat{\underline{e}}$ are independent \Leftrightarrow
 $\underline{P}\underline{y}$ and $(\underline{I} - \underline{P})\underline{y}$ are independent

use the previous result with $\underline{x} = \underline{y}$,

$$\underline{B}_1 = \underline{P} \text{ and } \underline{B}_2 = (\underline{I} - \underline{P})$$

$$\underline{B}_1 \underline{V} \underline{B}_2^T = \underline{P} \underline{V} \underline{I} (\underline{I} - \underline{P})^T = \underline{V} (\underline{P} - \underline{P}^2) = \underline{0}$$

$\Rightarrow \underline{P}\underline{y}$ and $(\underline{I} - \underline{P})\underline{y}$ are independent.

$\textcircled{7}$ Note that when \underline{x} is full rank

$$\hat{\underline{\beta}} = \underbrace{(\underline{x}'\underline{x})^{-1} \underline{x}'\underline{y}}_{\underline{B}_1} \sim N(\underline{B}_1 \underline{\mu}, \underline{B}_1 \underline{V} \underline{B}_1^T)$$

$\textcircled{8}$ $\underline{y} \sim N(\underline{\mu}, \underline{V})$ [and in our case $\underline{\mu} = \underline{x}\underline{\beta}$
 and $\underline{V} = \underline{I}$]

$$= N\left(\underline{x}'\underline{x}^{-1} \underline{x}'\underline{x}\underline{\beta}, \underline{V} (\underline{x}'\underline{x})^{-1} \underline{x}'\underline{x} (\underline{x}'\underline{x})^{-1}\right)$$

$$= N\left(\underline{\beta}, \underline{V} (\underline{x}'\underline{x})^{-1}\right)$$

$\textcircled{9}$ what is the distribution of $\underline{\lambda}^T \hat{\underline{\beta}}$ when $\underline{\lambda}^T \underline{\beta}$ is estimable?

$$\underline{\lambda}^T \hat{\underline{\beta}} = \underline{\lambda}^T (\underline{x}'\underline{x})^{-1} \underline{x}'\underline{y} \sim N\left(\underline{\lambda}^T \underline{\beta}, \underline{\lambda}^T (\underline{x}'\underline{x})^{-1} \underline{\lambda} \underline{V}\right)$$

$\textcircled{10}$ Def: let \underline{x} be a p -dimensional random vector and let \underline{A} be a $p \times p$ symmetric matrix. A quadratic form is a random variable defined by $\underline{x}^T \underline{A} \underline{x}$

We are interested to find distributions of quadratic forms.

$$\underline{y}^T \underline{y} = \underline{\hat{y}}^T \underline{\hat{y}} + \underline{\hat{e}}^T \underline{\hat{e}}$$

$$\underline{\hat{y}}^T \underline{\hat{y}} = [(\underline{I} - \underline{P}) \underline{y}]^T [(\underline{I} - \underline{P}) \underline{y}] = \underline{y}^T (\underline{I} - \underline{P})^T \underline{y} = \underline{y}^T (\underline{I} - \underline{P}) \underline{y}$$

$$\underline{\hat{e}}^T \underline{\hat{e}} = \underline{y}^T (\underline{I} - \underline{P}) \underline{y}$$

Recall: Let $\underline{z} \sim N_p(\underline{0}, \underline{I}_p)$ then $u = \underline{z}^T \underline{z} = \sum_{j=1}^p z_j^2$ follows χ^2 distribution with p -degrees of freedom.
 $u \sim \chi^2(p)$.

Def: z_1, \dots, z_p are s.t. $z_i \sim N(\mu_i, 1)$, z_i 's are independent.
 What is the distribution of $u = \sum_{i=1}^p z_i^2$.

$\sum_{i=1}^p z_i^2$ follows a noncentral χ^2 with p degrees of freedom and a noncentrality parameter $\phi = \sum_{i=1}^p \frac{\mu_i^2}{2}$. We write $u \sim \chi^2(p, \phi)$

Result: $x \sim \chi^2(\kappa, \phi)$ and $y \sim \chi^2(s, \delta)$ with x and y independent $\Rightarrow x+y \sim \chi^2(\kappa+s, \phi+\delta)$.

$\underline{x} \sim N_p(\underline{\mu}, \underline{I}) \Rightarrow \underline{x}^T \underline{x} \sim \chi^2(p, \underline{\mu}^T \underline{\mu} / 2)$.

Result: If $u \sim \chi^2(p, \phi)$, then $P(u > c)$ is strictly increasing in ϕ for fixed p and $c > 0$.

Result: Let \underline{x} be a random vector with $\underline{x} \sim N(\underline{\mu}, \underline{I})$ and \underline{M} is a ~~proj~~ projection matrix (perpendicular), then $\underline{x}^T \underline{M} \underline{x} \sim \chi^2(\text{rank}(\underline{M}), \underline{\mu}^T \underline{M} \underline{\mu} / 2)$.

Result: $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$, let \underline{A} be a symmetric matrix. Then if $\underline{A} \underline{V}$ is idempotent with rank s , then $\underline{x}^T \underline{A} \underline{x} \sim \chi^2(s, \underline{\mu}^T \underline{A} \underline{\mu} / 2)$.

Result: Let $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$ and let \underline{A} be a symmetric matrix with rank s ; then if $\underline{B} \underline{V} \underline{A} = \underline{0}$ then $\underline{B} \underline{x}$ and $\underline{x}^T \underline{A} \underline{x}$ are independent.

Cor: ~~where~~ $\underline{x} \sim N_p(\underline{\mu}, \underline{V})$ and $\underline{A}, \underline{B}$ are symmetric matrices. \underline{A} is of rank s and \underline{B} is of rank s . Then $\underline{B} \underline{V} \underline{A} = \underline{0}$ implies $\underline{x}^T \underline{A} \underline{x}$ and $\underline{x}^T \underline{B} \underline{x}$ are independent.

Goal: Identify distribution of $\frac{\hat{\underline{y}}^T \hat{\underline{y}}}{\underline{v}}$, $\frac{\hat{\underline{e}}^T \hat{\underline{e}}}{2\underline{v}}$ and so on.

$$\underline{y} \sim N(\underline{X} \underline{\beta}, \underline{v} \underline{I}) \Rightarrow \frac{\underline{y}}{\underline{v}} \sim N\left(\frac{\underline{X} \underline{\beta}}{\underline{v}}, \underline{I}\right)$$

$$\left(\frac{\underline{y}}{\underline{v}}\right)^T \left(\frac{\underline{y}}{\underline{v}}\right) \sim \chi^2\left(n, \frac{(\underline{X} \underline{\beta})^T (\underline{X} \underline{\beta})}{2\underline{v}}\right) = \chi^2\left(n, \frac{\underline{\beta}^T \underline{X}^T \underline{X} \underline{\beta}}{2\underline{v}}\right)$$

$$\Rightarrow \frac{\underline{y}^T \underline{y}}{\underline{v}^2} \sim \chi^2\left(n, \frac{\underline{\beta}^T \underline{X}^T \underline{X} \underline{\beta}}{2\underline{v}}\right)$$

$$\begin{aligned} \left(\frac{\underline{y}^T}{\underline{v}}\right) \underline{P} \left(\frac{\underline{y}}{\underline{v}}\right) &\sim \chi^2\left(\text{rank}(\underline{X}), \frac{(\underline{X} \underline{\beta})^T \underline{P} (\underline{X} \underline{\beta})}{2\underline{v}}\right) \\ &= \chi^2\left(\text{rank}(\underline{X}), \frac{\underline{\beta}^T \underline{X}^T \underline{X} \underline{\beta}}{2\underline{v}}\right) \end{aligned}$$

$$\begin{aligned} \left(\frac{\underline{y}}{\sqrt{\sigma^2}}\right)^T (I - \underline{P}) \left(\frac{\underline{y}}{\sqrt{\sigma^2}}\right) &\sim \chi^2(n - \text{rank}(\underline{X}), \underbrace{(\underline{X}\beta)^T (I - \underline{P}) \underline{X}\beta}_{2\sigma^2}) \\ &= \chi^2(n - \text{rank}(\underline{X}), 0) \\ &= \chi^2(n - \text{rank}(\underline{X})) \end{aligned}$$

Recall: The unbiased estimator of σ^2 is

$$\frac{\underline{y}^T (I - \underline{P}) \underline{y}}{\sigma^2 (n - \text{rank}(\underline{X}))} \sim \frac{\chi^2(n - \text{rank}(\underline{X}))}{n - \text{rank}(\underline{X})}$$

Question: Are $\underline{y}^T \underline{P} \underline{y}$ and $\underline{y}^T (I - \underline{P}) \underline{y}$ independent?

~~Yes~~ Yes, they are independent as

$$\underline{P} \sigma^2 I (I - \underline{P}) = \underline{0}$$

What is the distribution of $\frac{\|\hat{\underline{y}}\|^2 / \text{rank}(\underline{X})}{\|\hat{\underline{e}}\|^2 / (n - \text{rank}(\underline{X}))}$

$$= \frac{\underline{y}^T \underline{P} \underline{y} / \text{rank}(\underline{X})}{\underline{y}^T (I - \underline{P}) \underline{y} / (n - \text{rank}(\underline{X}))}$$

Note that,

$$\frac{\underline{y}^T (I - \underline{P}) \underline{y}}{\sigma^2} \sim \chi^2(n - \text{rank}(\underline{X})),$$

$$\underline{y}^T \underline{P} \underline{y} \sim \chi^2(\text{rank}(\underline{X}), \frac{\beta^T \underline{X}^T \underline{X} \beta}{2\sigma^2})$$

Df: Let u_1 and u_2 be independent random variables, with $u_1 \sim \chi^2(p_1)$ and $u_2 \sim \chi^2(p_2)$, then $f = \frac{u_1/p_1}{u_2/p_2}$ follows an F-distribution with p_1, p_2 degrees of freedom, denoted by $f \sim F(p_1, p_2)$

Def: Let u_1 and u_2 be independent random variables, with $u_1 \sim \chi^2(p_1, \phi)$ and $u_2 \sim \chi^2(p_2)$; then $f = \frac{u_1/p_1}{u_2/p_2}$ has the F-distribution with p_1 and p_2 degrees of freedom, noncentrality parameter ϕ , denoted by $f \sim F(p_1, p_2, \phi)$.

~~(Def)~~ ~~(1)~~ (2)

With this definition of non-central F, we conclude that,

$$\frac{\underline{y}^T \underline{P} \underline{y} / \text{rank}(\underline{X})}{\underline{y}^T (\underline{I} - \underline{P}) \underline{y} / (n - \text{rank}(\underline{X}))} \sim F(\text{rank}(\underline{X}), n - \text{rank}(\underline{X}), \frac{\underline{\beta}^T \underline{X}^T \underline{X} \underline{\beta}}{2})$$

Def: Let $u \sim N(\mu, 1)$ and $v \sim \chi^2(k)$. If u and v are independent then $t = \frac{u}{\sqrt{v/k}}$ has the noncentered Student's t-distribution with k degrees of freedom and a noncentrality parameter μ , denoted by $t \sim t(k, \mu)$. If $\mu = 0$, the distribution is generally known as Student's t, denoted by $t \sim t(k)$.

Why is it important?

Let $\underline{\lambda}^T \underline{\beta}$ be any estimable function.

$\underline{\lambda}^T \hat{\underline{\beta}}$ is the unbiased estimator of $\underline{\lambda}^T \underline{\beta}$.

back (1)

$$\underline{\lambda}^T \hat{\underline{\beta}} \sim N(\underline{\lambda}^T \underline{\beta}, \sigma^2 \underline{\lambda}^T (\underline{X}'\underline{X})^{-1} \underline{\lambda})$$

$$\frac{\underline{\lambda}^T \hat{\underline{\beta}} - \underline{\lambda}^T \underline{\beta}}{\sqrt{\left\{ \frac{\underline{y}^T (\underline{I} - \underline{P}) \underline{y}}{n - \text{rank}(\underline{X})} \right\} \underline{\lambda}^T (\underline{X}'\underline{X})^{-1} \underline{\lambda}}} = \frac{\underline{\lambda}^T \hat{\underline{\beta}} - \underline{\lambda}^T \underline{\beta}}{\sqrt{\sigma^2 \underline{\lambda}^T (\underline{X}'\underline{X})^{-1} \underline{\lambda}}}$$

$$\sqrt{\frac{\underline{y}^T (\underline{I} - \underline{P}) \underline{y}}{\sigma^2 (n - \text{rank}(\underline{X}))}}$$

$$\frac{\underline{\lambda}^T \hat{\underline{\beta}} - \underline{\lambda}^T \underline{\beta}}{\sqrt{\sigma^2 \underline{\lambda}^T (\underline{X}'\underline{X})^{-1} \underline{\lambda}}} \sim N(0, 1)$$

$$\frac{\underline{y}^T (\underline{I} - \underline{P}) \underline{y}}{\sigma^2} \sim \chi^2 (n - \text{rank}(\underline{X}))$$

$$\sqrt{\quad} \sim t(n - \text{rank}(\underline{X}))$$

$$\Leftrightarrow \frac{\underline{\lambda}^T \hat{\underline{\beta}}}{\sqrt{\left\{ \frac{\underline{y}^T (\underline{I} - \underline{P}) \underline{y}}{n - \text{rank}(\underline{X})} \right\} \underline{\lambda}^T (\underline{X}'\underline{X})^{-1} \underline{\lambda}}} \sim t(n - k, \underline{\lambda}^T \underline{\beta})$$

One way Anova Model:

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad i=1, 2, 3, \dots, \text{ } \quad n_i = 4, 6, 6, \dots$$

$$\underline{y} = \begin{bmatrix} \underline{1}_4 & \underline{1}_4 & \underline{0}_4 & \underline{0}_4 \\ \underline{1}_6 & \underline{0}_6 & \underline{1}_6 & \underline{0}_6 \\ \underline{1}_6 & \underline{0}_6 & \underline{0}_6 & \underline{1}_6 \\ \text{...} & \text{...} & \text{...} & \text{...} \end{bmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \text{...} \end{pmatrix} + \underline{\varepsilon}$$

$$\underline{1}_k = (\underbrace{1, \dots, 1}_{k \text{ times}})'$$

back ②

$$\underline{X} = \begin{bmatrix} \underline{1}_4 & \underline{1}_4 & \underline{0}_4 & \underline{0}_4 \\ \underline{1}_6 & \underline{0}_6 & \underline{1}_6 & \underline{0}_6 \\ \underline{1}_6 & \underline{0}_6 & \underline{0}_6 & \underline{1}_6 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_N$

$$\underline{P}_W = \underline{W} (\underline{W}^T \underline{W})^{-1} \underline{W}^T = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \frac{1}{16} (1, \dots, 1)$$

$$= \frac{1}{16} J_{16}$$

$$C(\underline{W}) \subset C(\underline{X})$$

the projection matrix onto the orthogonal space of $C(\underline{W})$ in $C(\underline{X})$, $\underline{P}_X - \underline{P}_W$

$$\underline{P}_X =$$